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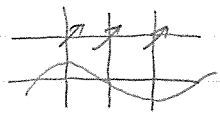
9/8 Anomaly, Generalized symmetry

QFT ← Lattice system, String theory } UV complete

- 3d Ising model ← quantum magnetization



↓ low energy limit

3d real scalar theory (continuum QFT)

$\mathcal{L} = (\partial_\mu \varphi)^2 + m^2\varphi^2 + \lambda\varphi^4 + \cdots$

↑ continuum field φ

Quite often, QFT is strongly coupled → we don't know explicit solutions

Symmetry — anomaly

e.g.) 3d Ising $\varphi(x) : \mathbb{R}^3 \to \mathbb{R}$ (space (spacetime) ← target space)

$\mathcal{L} = (\partial_\mu \varphi)^2 + m^2\varphi^2 + \lambda\varphi^4$ ← function of φ^2!

sym : $\varphi \to -\varphi$ $\quad \mathcal{L}(-\varphi) = \mathcal{L}(\varphi)$

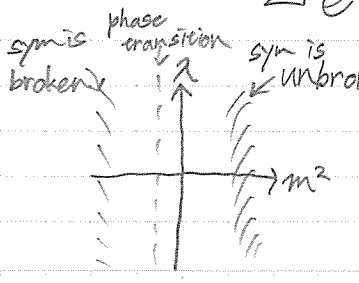
Questions : · Is this sym spontaneously broken?

$Z = \sum e^{-E(\uparrow\uparrow\downarrow\uparrow)}$ → ground state

2 vacua { ↑↑↑↑ $\langle S \rangle = \uparrow$ spontaneous broken s
↓↓↓↓ $\langle S \rangle = \downarrow$

Ferromagne

↑↓↑↓ $\langle S \rangle = 0$ ← sym is unbroken



phase transition

◦ Generalized symmetry

· p-form sym. → topological order 4d YM-theory

· n-group sym. (= mixture of different p-form sym.)

· (for 1+1 dim QFT) fusion-category sym.

◦ Anomaly

↳ d-dim QFT has a global sym G

⇝ We can ask if we can gauge this sym.

G-gauge field A

$$Z_{\text{gauged}} = \int \mathcal{D}A\, e^{-S_{YM}[A]} \cdot Z_{QFT}[A]$$

if $Z_{QFT}[A+\delta_\lambda A] = Z_{QFT}[A]$ (↳ gauge trans.), then we can gauge the sym.

However, we often encounter

$$Z_{QFT}[A+\delta_\lambda A] = Z_{QFT}[A]\exp(i\mathcal{A}[A,\lambda])$$

('t Hooft anom. matching を使うには この形の(pure phase) anom. でないとダメ)

phase ambiguity

↑

(gauge) anomaly ('t Hooft) : (for gauge field) (for external field)

→ We cannot gauge the sym. (in d-dim QFT)

Theorem ('t Hooft anomaly matching)

Anomaly $\mathcal{A}$ is RG-invariant. ⌟ (does not change unbroken any G-symmetric local and continuous deformation of the QFT)

(if $G \to$ SSB : WZW term を入れた eff. action を考えて 't Hooft anom. matching
e.g.) QCD ~~$SU(N_F) \times SU(N_F)$~~ : π action w/ WZW term)

Quantum mechanics of single spin S

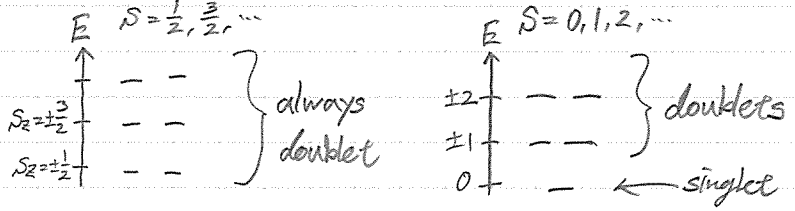
← $\begin{cases} S = \frac{1}{2}, \frac{3}{2}, \cdots & \text{anomaly} \to \text{state degeneracy} \\ S = 0, 1, \cdots & \text{no-anomaly} \to \text{singlet state} \end{cases}$

$\hat{S}_x, \hat{S}_y, \hat{S}_z$ $\quad \hat{S}^2 = \sum_{i=1}^{3} \hat{S}_i^2 = S(S+1)$

Hamiltonian : $\hat{H} = J\hat{S}_z^2$

$\mathcal{H} = \bigoplus_{S_z=-S}^{S} \mathbb{C}\,|S_z\rangle$ $\quad (\dim \mathcal{H} = 2S+1)$

$\hat{H}|S_z\rangle = \underbrace{JS_z^2}_{\text{energy}}|S_z\rangle$ $\quad S = 0, 1, 2, \cdots$



Sym. $J = 0$, $G = SO(3)$

Note : $G \neq SU(2)$ even for $S = \frac{1}{2}, \frac{3}{2}, \cdots$

Symmetry := Transformation of physical state , which does not E

{physical states} $\longleftrightarrow$ $\mathcal{H} / \mathbb{C}^\times$ ← $(\mathbb{C}/\{0\})$
~~~ overall phase cannot be detectable

$-1 \in SU(2)$
~~~ 2π-rotation of spin

$|S_z\rangle \longrightarrow \underbrace{e^{i2S_z}}_{\text{for all } S_z \text{ if } S = \frac{1}{2}, \frac{3}{2}, \cdots}|S_z\rangle$ $\quad \to G = SU(2)/\underbrace{\{1, -1\}}_{= \mathbb{Z}_2}$

$= SO(3)$ //

$\uparrow$ this is an overall phase, which is "unphysical"

4.

e.g.) $J>0$, $H = J S_z^2$

$G = O(2) = SO(2) \rtimes \mathbb{Z}_2 \subset SO(3)$ ← C: charge conjugation

$\begin{pmatrix} S_x \\ S_y \end{pmatrix}$ ↻ rotational sym. $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \\ & & 1 \end{pmatrix} : SO(2)$

$S_z \to -S_z$ $\det \underbrace{\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}}_{\in SO(3)} = -1$ ← ダメ

↓ 同時に $S_y \to -S_y$ $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$: generation of $\mathbb{Z}_2$ in $SO(3)$

(when we consider "parity", $G = SO(3) \times (\mathbb{Z}_2)_P$ ← parity

$G = O(2) \times (\mathbb{Z}_2)_P = (SO(2) \rtimes \mathbb{Z}_2) \times (\mathbb{Z}_2)_P \subset SO(3) \times (\mathbb{Z}_2)_P$

↑ $\det \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = -1 \notin SO(3)$

now, we neglect parity (because there is no interesting things))

$G = SO(2) \rtimes \mathbb{Z}_2$ has an anomaly if $S = \frac{1}{2}, \frac{3}{2}, \cdots$

∴ does not have ∴ $S = 0, 1, 2, \cdots$

Let us consider $\underbrace{SO(2)}_{\hat{g}_\alpha = e^{i\alpha \cdot \hat{S}_z}}$-twisted partition function

$$Z(\alpha) = \mathrm{tr}\left[\hat{g}_\alpha e^{-\beta \hat{H}}\right] = \sum_{S_z=-S}^{S} e^{i\alpha S_z} e^{-\beta J S_z^2}$$

↑

$Z(\alpha)$ is inv. under C : $Z(\alpha) \xmapsto{C} Z(-\alpha)$

$$= \sum_{S_z=-S}^{S} e^{i(-\alpha)\underbrace{(-S_z)}_{=S_z'}} e^{-\beta J \underbrace{S_z^2}_{=(S_z')^2}}$$

$$= \sum_{S_z=-S}^{S} e^{i\alpha S_z} e^{-\beta J S_z^2}$$

$$= Z(\alpha)$$

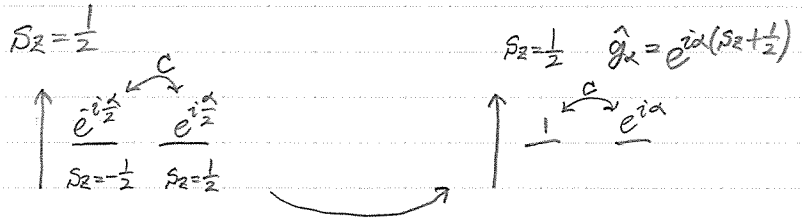
However, $Z(\alpha)$ is not $SO(2)$-inv.

$$\left(Z(\alpha) \overset{\text{naive}}{=}\right) Z(\alpha+2\pi) = \sum_{S_z=S}^{S} e^{i(\alpha+2\pi)S_z} e^{-\beta J S_z^2}$$

$$= e^{2\pi i S_z} e^{i\alpha S_z} \to \begin{cases} +1 & S = 0, 1, \cdots \\ -1 & S = \frac{1}{2}, \frac{3}{2} \cdots \end{cases}$$

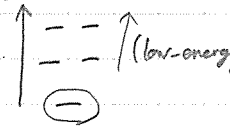
$$= \begin{cases} Z(\alpha) & S_z = 0, 1, 2, \cdots \\ -Z(\alpha) & S_z = \frac{1}{2}, \frac{3}{2}, \cdots \end{cases}$$

↑ anomaly



Anomaly matching

- for $S = 0, 1, 2, \cdots$ → No-anomaly



→ For $J>0$, the ground state is singlet

$|0\rangle \overset{O(2)}{\longmapsto} |0\rangle$

$\Rightarrow \langle \mathcal{O} \rangle = \langle 0 | \mathcal{O} | 0 \rangle$ → No-phase change in $\langle \mathcal{O} \rangle$ by $O(2)$ trans.

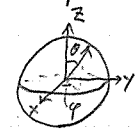
$\Rightarrow$ No-anomaly

- for $S = \frac{1}{2}, \frac{3}{2}, \cdots$ → Anomaly

: Always doublets

Rederivation of anomaly in view of path-int.

$\vec{S} = S \cdot \vec{n}$ ← unit vector



$\begin{cases} 0 \le \theta \le \pi \\ \varphi \sim \varphi + 2\pi \end{cases}$

$|S| \gg 1$

$$S_E = i \underbrace{\int S(1-\cos\theta)\,d\varphi}_{\text{WZ term}} + \int d\tau \left(\dot{\vec{n}}^2 + JS^2 \underbrace{n_z^2}_{(\cos\theta)^2} \right)$$

Motivation : $e^{iS(1-\cos\theta)\varphi}$ (WZ term)

$\to \varphi \sim \varphi + 2\pi \Rightarrow S(1-\cos\theta) = n \in \mathbb{Z}$

$\to 0 \le 1-\cos\theta = \frac{n}{S} \le 2$ $\quad n = 0, 1, 2, \cdots, 2S$: $(2S+1)$ states

$H = JS_z^2$ $\quad G = \underbrace{SO(2)}_{\varphi \mapsto \varphi + \alpha} \times \mathbb{Z}_2$ $\begin{cases} \theta \mapsto \pi - \theta \\ \varphi \mapsto -\varphi \end{cases}$

Invariance of WZ term under $\mathbb{Z}_2$

$$i\int S(1-\cos\theta)\,d\varphi$$

$$\xmapsto{\mathbb{Z}_2} i\int S \underbrace{(1-\cos(\pi-\theta))}_{1+\cos\theta} \underbrace{d(-\varphi)}_{-d\varphi} = i\int S(1-\cos\theta)\,d\varphi - i(2S)\int d\varphi$$

($\int$: int., 2π-int.)

$\equiv 0 \pmod{2\pi i}$

↙ $Z = e^{-S_E}$ としてしまえば効かない ⇒ $\mathbb{Z}_2$ inv.

In order to detect $\underbrace{O(2)}_{= SO(2) \times \mathbb{Z}_2}$ anom., we introduce

A : $U(1)$ ($\supset SO(2)$) gauge field $\quad A \mapsto A + d\lambda$, $\varphi \mapsto \varphi - \lambda$

$\Rightarrow d\varphi \rightsquigarrow (d\varphi + A) = D\varphi$

$$S_{\text{gauged}} = i\int S(1-\cos\theta)\,(d\varphi + A) + \cdots$$

$$\xmapsto{\mathbb{Z}_2} i\int S \underbrace{(1-\cos(\pi-\theta))}_{1+\cos\theta} \underbrace{(d(-\varphi) - A)}_{-D\varphi} = i\int S(1-\cos\theta)(d\varphi + A) - i(2S)\int (d\varphi + A)$$

phase が出た ↓ (2π-int.) ← 落ちない!

$$Z[A] = \int \mathcal{D}\varphi \mathcal{D}\theta \, e^{-S_{gauged}}$$

$$\xrightarrow{\mathbb{Z}_2} \int \mathcal{D}\varphi \mathcal{D}\theta \, e^{-S_{gauged}} \times \underbrace{e^{i(2S)\int A}}_{\text{indep of dynamical fields}}$$

$$= \underbrace{e^{i(2S)\int A}}_{\text{anomalous phase?}} Z[A]$$

Situation · We introduced $A : U(1)$ gauge field

"$Z[A]$" by replacing $d\varphi \rightsquigarrow D\varphi = d\varphi + A$

By def. $Z[A+d\lambda] = Z[A]$

· However, $(\mathbb{Z}_2)_C : Z[A] \mapsto e^{i(2S)\int A}_{\text{"}\alpha\text{"}} Z[A]$

We can still try our best to cancell this "anomaly"

Local counter term of A w/ $U(1)$ gauge inv. w/ $k \in \mathbb{Z}$

$$k\,CS[A] = ik\int A$$

1d Chern-Simons

$$ik\,CS[A+d\lambda] = ik\int (A+d\lambda) \bmod 2\pi i$$

$$= ik\,CS[A] + \underbrace{ik}_{\text{int.}} \underbrace{\int d\lambda}_{2\pi\text{-int.}}$$

"$Z_k[A] = e^{ik\int A} Z[A]$"

$$Z_k[A] = e^{ik\int A} Z[A] \xrightarrow{\mathbb{Z}_2} e^{-ik\int A} e^{i(2S)\int A} Z[A]$$

$$= e^{i(2S-k)\int A} Z[A]$$

If $S = 0, 1, 2, \cdots$ we can choose $k = 2S$,

So that $\underset{k=2S}{Z}[A] \xrightarrow{\mathbb{Z}_2} \underset{k=2S}{Z}[A]$ $\Rightarrow$ No anom.

For $S=\frac{1}{2},\frac{3}{2},\cdots$, we cannot choose $k=2S$ for $U(1)$ inv.

$\to Z_k[A]$ must violate C $\Rightarrow$ anomaly

$\rightsquigarrow$ We "rederived" the anom. of $O(2)$ sym.

comment. $J\to\infty$ $H\sim JS^2\cos^2\theta$ $\Rightarrow\theta=\frac{\pi}{2}$ dynamics is localized

on S^1

$\to$ 1 dim QM (on S^1)

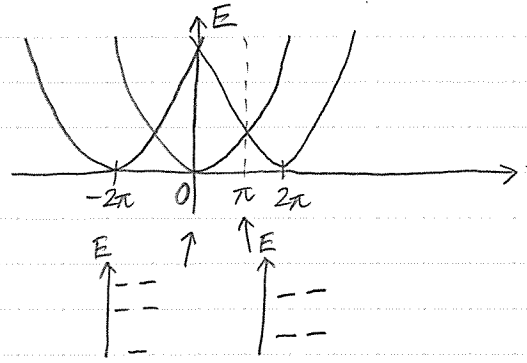
$S=\int|d\varphi|^2+i\frac{\Theta}{2\pi}\int d\varphi$

↑ WZ term $\Theta=\frac{\pi}{2}$

$E=\Theta\left(n+\frac{\Theta}{2\pi}\right)^2$

$\Theta=2\pi S\in\pi\mathbb{Z}$ $\begin{cases}=\pi & \text{for } S=\frac{1}{2},\frac{3}{2},\cdots\\ =0 & \text{for } S=0,1,2,\cdots\end{cases}$ $(\mathrm{mod}\ 2\pi)$

$G=SO(2)\times\mathbb{Z}_2\to\mathbb{Z}_{2n}$



※ path-int の例で $A\sim$ みたいに取れば前半の twist した例になっている

$\int A\sim\alpha$ $\alpha\mapsto\alpha+2\pi$

$\Updownarrow$ で対応している

$A\mapsto A+d\lambda$, $\int d\lambda=2\pi$

※ 何故 CS level (k)は int?

$e^{ik\int A} \mapsto e^{ik\int(A+d\lambda)}$ $\quad d\lambda = g dg^{-1}$ $\quad g: S^1 \mapsto U(1)$ $\quad (0\sim\beta) \quad (0\sim 2\pi)$

$\int d\lambda = \lambda(\beta) - \lambda(0) = 2\pi n$

$\to e^{ik\cdot\int d\lambda} = e^{ik\cdot 2\pi n} \overset{!}{=} 1$ $\quad \Rightarrow k$ は int.

↑ gauge inv. に成って欲しい

-PM① -

QFT ← We want to know low energy physics w/o solving it explicitly

Strategies
- We identify the sym. correctly
- We introduce the bg gauge field A, "$Z[A]$"
- We check if it is gauge inv. $\Leftrightarrow$ anomaly

$\rightsquigarrow$ Anomaly matching can constrain possibility of low energy spectrum

QM of single spin S, $H = J_1 S_z^2$

$G = \underbrace{SO(2) \times \mathbb{Z}_2}_{\| \; O(2)} \subset SO(3)$

$A: \underset{\| \; U(1)}{SO(2)}$ gauge field

$Z[A] = \int \mathcal{D}\varphi \mathcal{D}\theta \, e^{-S}|_{d\varphi \to d\varphi + A}$ + local counter terms

$\Leftrightarrow \begin{cases} S = 0, 1, \cdots & \text{No anomaly} \to \text{singlet G.S.} \\ S = \frac{1}{2}, \frac{3}{2}, \cdots & \text{Anomaly} \to \text{doublet} \end{cases}$

d-dim QFT (relativistic, Euclidean)

Def. (Global sym.)

d-dim QFT has global sym. G ← compact grp.

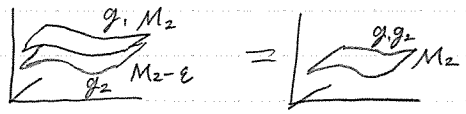
$\overset{\text{def.}}{\Leftrightarrow}$ X : d-dim Rieman mfd. (= Euclidean spacetime)

We assign an operator $U(g)(M_{d-1})$

for each $\begin{cases} M_{d-1} \subset X & (d-1 \text{ dim codim-1 mfd}) \\ g \in G \end{cases}$

$\{U_g(M_{d-1})\}$ satisfies

- (Group law) $U_{g_1}(M_{d-1})U_{g_2}(M_{d-1}) = U_{g_1 g_2}(M_{d-1})$



- (Conservation law) $U_g(M_{d-1})$ is topological,

i.e. $\langle U_g(M_{d-1})\, \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots \rangle$

$= \langle U_g(M_{d-1}+\delta)\, \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots \rangle$

↑ infinitesimal deformation

- (Transformation law)

Local op.s fall into a rep of G : $V_i(x)$

$U_g(S_0^{d-1})\, V_i(0) = \exists R(g)_i{}^j V_j(0)$

↑ small space around $x=0$; ↑ a rep. of G

- (Faithful transformation)

For some $V_i(0) \neq 0$, its $R(g)$ is faithful. (i.e. $\forall g \neq 1$, $R(g) \neq 1$)

Continuous Sym.

QFT $S[\varphi]$ has a continuum sym. G

G: Lie group (continuous) $\quad U(1), SU(N), SO, \cdots$

$\varepsilon \in \mathrm{Lie}(G) \rightsquigarrow \varphi + \delta_\varepsilon \varphi$ infinitesimal transformation

$S[\varphi + \delta_\varepsilon \varphi]$

↑ const. in spacetime

Noether's thm. $\to$ Noether current $j^a_\mu \quad \partial_\mu j^a_\mu = 0$

Quick derivation: $\varepsilon \mapsto \varepsilon(x)$ infinitesimal, but depend on x

$$0 = \int \mathcal{D}\varphi \left(e^{-S[\varphi + \delta_{\varepsilon(x)}\varphi]} - e^{-S[\varphi]} \right)$$

↑ if ε is const.

$$= \int \mathcal{D}\varphi \, e^{-S[\varphi]} \cdot \underbrace{\left(\int_x \delta_\varepsilon \varphi(x) \cdot \frac{\delta S}{\delta \varphi(x)} \right)}_{d\varepsilon^a \wedge j^a \leftarrow d-1 \text{ form}} \quad \text{up to EOM}$$

$$j^a = \frac{1}{(d-1)!} \varepsilon_{\mu_1 \mu_2 \cdots \mu_d} j^a_{\mu_1} \, dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_d}$$

$$dj^a = 0 \qquad (\Leftrightarrow \partial_\mu j^a_\mu = 0)$$

$$U_{\varepsilon^a}(M_{d-1}) = \exp\left(i\varepsilon^a \int_{M_{d-1}} j^a \right) \qquad X = \underbrace{\mathbb{R}^{d-1}}_{\text{space}} \times \underbrace{\mathbb{R}}_{\text{time}}$$

$$\left(\text{if } M_{d-1} = \underbrace{\mathbb{R}^{d-1}}_{\text{space}}, \; U_{\varepsilon^a}(\mathbb{R}^{d-1}) = e^{i\varepsilon^a \underbrace{\int j^a_0 \, d^{d-1}x}_{= Q^a}} \right)$$

$$[Q^a, \varphi(x)] = \delta_a \varphi(x)$$

· conservation law $\quad U(\mathbb{R}^{d-1})(t) = U(\mathbb{R}^{d-1})(0)$

add an infinitesimal def. $\delta M_{d-1} = \partial V_d$

$$U(M_{d-1}) = U(M_{d-1}) \cdot e^{i\varepsilon^a \underbrace{\int_{V_d} dj}_{\int_{V_d} d\tilde{j} = 0}}$$

SSB

Sym. G of QFT is spontaneously broken

$\overset{\text{def.}}{\Leftrightarrow}$ $\exists$ local op. $V_i(x)$ in a nontrivial rep of G

s.t. $\langle V_i(x)^* V_j(0)\rangle \not\to 0$ as $|x|\to\infty$, $\mathrm{vol}(X)\to\infty$

(Usual def)
We add a "small" breaking term $S \mapsto S + \delta_\varepsilon S$, $\delta_\varepsilon S = \int_X \varepsilon V_j(x)$

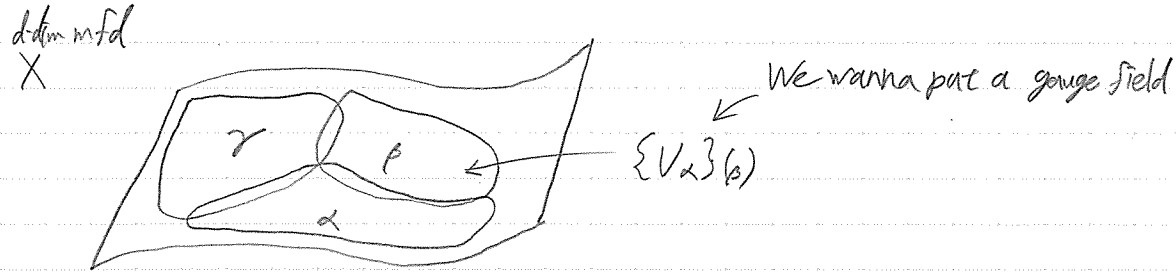
$\lim_{\delta_\varepsilon S\to 0} \lim_{\mathrm{vol}(X)\to\infty} \langle V_i(x)\rangle_{\delta_\varepsilon S} \neq 0$

Gauging procedure

d-dim QFT w/ sym. G

$\{U_g(M_{d-1})\}$, $g \in G$

we want to gauge G $\longrightarrow$ principal G-bdl We impose bdl inv.



$\{A_\alpha$: Lie(G)-valued 1-form on U_α
$g_{\alpha\beta}$: G-valued function on $U_\alpha \cap U_\beta$

↑ Nice gauge properties (on $U_\alpha \cap U_\beta \cap U_\gamma$) $g_{\gamma\beta} g_{\beta\alpha} g_{\alpha\gamma} = 1$

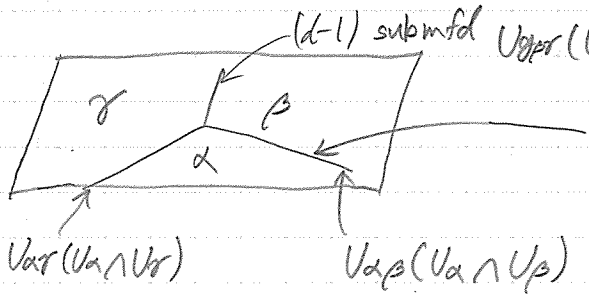
$$A_\beta|_{U_\alpha \cap U_\beta} = (g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta} d g_{\alpha\beta})|_{U_\alpha \cap U_\beta}$$

$$\psi_\beta|_{U_\alpha \cap U_\beta} = R(g_{\alpha\beta}) \cdot \psi_\alpha$$

A: G-gauge field $= \text{Data}\{A_\alpha\}, \{g_{\alpha\beta}\}$

- for "usual" continuous sym, this is th "usual" gauging.
- This is valid for other sym.s
  - "discrete" $\mathbb{Z}_n, S_n, A_n, \cdots$
  - "disconnected components"

$A_\alpha = 0 \Rightarrow$ All properties of discrete gauge fields are induced in $\{g_{\alpha\beta}\}$



't Hooft anom.

Def. d-dim QFT w/ sym. G

sym. G has an 't Hooft anom.

$\overset{\text{def.}}{\Leftrightarrow}$ A : bg G-gauge field

$A \mapsto A + \delta_\lambda A$: G-gauge trans.

$$Z[A] \mapsto Z[A + \delta_\lambda A] = Z[A]\, e^{i \int \mathcal{A}[\lambda, A]}$$

s.t. $\mathcal{A}$ cannot be eliminated by local d-dim action of A

QM examples : $G = O(2) \longrightarrow \mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$

$$\left\langle \; \overset{g_1}{\underset{g_2}{\bigcirc}} \; \right\rangle = e^{i\omega(g_1,g_2)} \left\langle \; \bigcirc^{\rho(g_1,g_2)} \; \right\rangle$$

↑ projective phase

↑ G-gauge field imaginally-time

$\rho(g_1) \rightsquigarrow \rho(g_1) e^{i\omega(g_1)}$

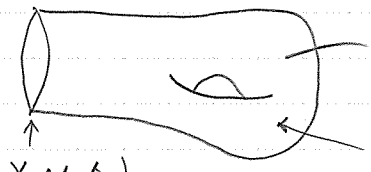
$\rho(g_1,g_2) \to \omega(g_1,g_2) + \underbrace{\alpha(g_1)+\alpha(g_2)-\alpha(g_1,g_2)}_{\delta\alpha(g_1,g_2)}$

If any $\alpha : G \mapsto U(1)$ cannot eliminate ω, this is G-anom.

Anomaly matching and Anomaly inflow

· (Anomaly-inflow hypothesis)

Assume d-dim QFT w/ sym. G has an 't Hooft anom. $\mathcal{A}$

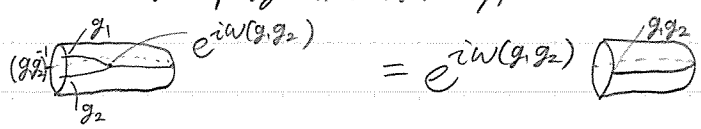


$e^{iS_{d+1}[A]}$

s.t. $\begin{cases} S_{d+1}[A+\delta_\alpha A] = S_{d+1}[A] - i\int_X \mathcal{A}(\alpha, A) & \text{mod } 2\pi i \\ \Rightarrow Z_{\text{local}}[A] = Z_{QFT,X}[A] \cdot e^{iS_{d+1}[A]} & \text{is gauge inv.} \end{cases}$

QM example

2-dim Dijkgraaf-Witten action satisfies anomaly-inflow hypothesis
(+ topological field theory)



Anomaly matching

Anomaly inflow hypothesis

$\to Z_{QFT}[A] \cdot e^{i S_{d+1,Y}[A]}$ is gauge inv.

($S_{d+1,Y}$: $(d+1)$ dim topological action; e^{iS}: anomaly coeff. takes discretised value)

$Z_{QFT}[A]$: G-sym. continious deformation of d-dim QFT does not change anoms.

$\rightsquigarrow$ Anomaly is RG inv.

-PM② -

4-dim YM $SU(N)$

- Lagrangian

$a \overset{local}{=} a_\mu^i(x) \underbrace{T^i}_{\text{Lie}(SU(N)) \text{ generators}} dx^\mu$

$f = da + i a \wedge a$: $SU(N)$ field strength

$$\mathcal{L} = \underbrace{\frac{1}{g^2} \mathrm{tr}(f \wedge *f)}_{\text{kinetic term}} + \underbrace{\frac{i\theta}{8\pi^2} \mathrm{tr}(f \wedge f)}_{\text{total derivative term}}$$

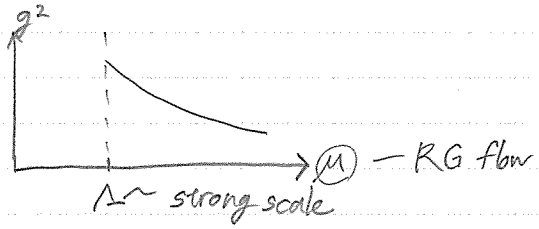
classical Lagrangian has a scale inv.

- gauge trans. : g : spacetime $\mapsto SU(N)$

$a \mapsto g^{-1} a g - i g^{-1} dg$

$f \mapsto g^{-1} f g$

- Known facts (beliefs)

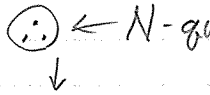


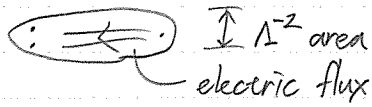
· Classical (Perturbative)

a : massless gluons $\Rightarrow$ Mass gap $\sim \Lambda$

· (Quark) Confinement

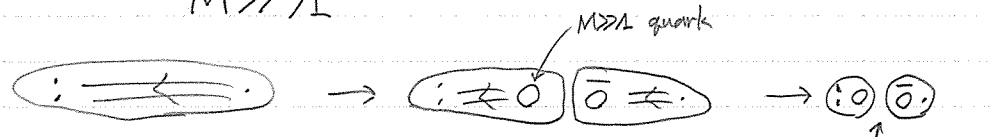
test quark = test particle in a fund. rep. of SU(N)





cf.) YM + massive dynamical (fund. in SU(N)) quark

$M \gg \Lambda$



If $M \gg \Lambda \rightsquigarrow$ For pair production $Q\bar{Q}$, we need $E \sim 2M$

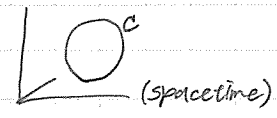
Total E of flux $\sim \Lambda^2 L$ ← length

However, usually E is distributed equally in L

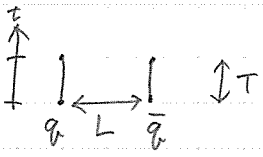
$\Rightarrow e^{-\# \frac{M}{\Lambda}}$ production rate

· { Wilson loop
Gapped phase of 4d YM classified by Wilson loops

Wilson loop : $W(C) = tr(P e^{i \oint_C a})$
closed loop

$=$ 

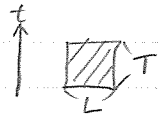
$\langle W(C) \rangle = e^{-T \cdot V_{q\bar{q}}(L)}$ (L × T)



Gapped phase by wilson

If $V_{q\bar{q}}(L) \propto \sigma \cdot L$ ($\sigma \sim \Lambda^{-2}$) $\Rightarrow$ confinement

$\Leftrightarrow \langle W(C) \rangle \sim e^{-\sigma T \cdot L}$: Area law confinement phase



Other possibilities

$\langle W(C) \rangle \sim e^{-\#(L+T)}$ ($\#$: $\sim \Lambda$)

perimeter (周長)

$\Rightarrow$ perimeter law (Deconfinement) Higgs phase

$\rightarrow$ We have different phases characterized by loop operators

not by local operators

Landau criterion

Q. Can we unify this Wilson criterion and the Landau criterion

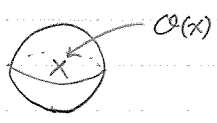
↓

A. We can do this by introducing a higher-form symmetry!

cf.) Ordinary sym : $U_{g \in G}(M_{d-1})$: a codim-1 topological operators

local op ↗ w/ the symmetry action

$g \in G$



$\rightarrow R(g) \cdot O(x)$

Def. (p-form sym.)

$U_g(M_{d-p-1})$ ← codim $(p+1)$ submanifold

$\uparrow G$

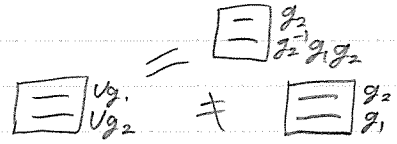
- $U_{g_1}(M_{d-p-1})\,U_{g_2}(M_{d-p-1}) = U_{g_1g_2}(M_{d-p-1})$
- $U_g(M_{d-p-1})$ is topological
- $V(C_p)$: operators defined on p-dim closed mfd

$$U_g(M_{d-p-1})\,V(C_p) = R(g)\,V(C_p)$$

- For sum V, $U_{g_1g_2}$ is a faithful rep. of G

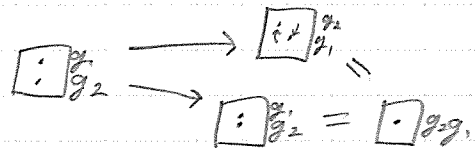
Note for $p \geq 1$, G is **Abelian**

for $p=0$,



$\Rightarrow$ non-abelian is possible

for $p \geq 1$,



4d $SU(N)$ YM has a $\mathbb{Z}_N$ 1-form sym.

(2章前半...)

○ exercise 4d $U(1)$ gauge theory coupled to charge-N Higgs field

a : $U(1)$ gauge field, φ : cpx scalar $\quad \varphi \mapsto e^{Ni\alpha}\varphi$: $U(1)$ sym is gauged

$$\Rightarrow \mathcal{L} = \frac{1}{2e^2} f \wedge *f + |(\partial_\mu + iNa_\mu)\varphi|^2 + V(\varphi^*\varphi)$$

- this theory has a $\mathbb{Z}_N$ 1-form sym.

Wilson loop $W(C)^k = e^{ki\oint_C a}$ $\quad C = L \times T$

q $\bar{q}$

Q. can we measure the gauge charge of q

→ In modulo N, we can measure the gauge charge q

codim-2 defect M_2

$$\langle U(M_2)\, W(C)^k \rangle = e^{\frac{2\pi i}{N}k} \langle W(C)^k \rangle$$

$$U(M_2) = e^{\frac{2\pi i}{N}\int_{M_2}\vec{E}\cdot d\vec{S}}$$

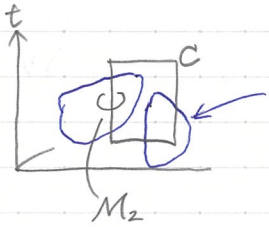
$k, k+N, k+2N, \cdots$

top. defect → $e^{\frac{2\pi i}{N}k}$

$$\Rightarrow \int_{M_2} E\, dS = k + N$$

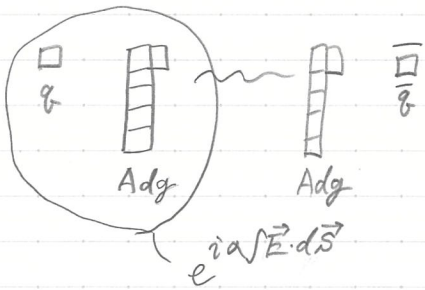
dynamical field's effect (Higgs)

ambiguous for $\mathbb{Z}_N$ 1-form sym.



dynamical field's world line $\sim W^N$

(4d SU(N) YM w/ $\mathbb{Z}_N$ 1-form sym. に戻る)



We cannot identify □ itself (rep. of q)

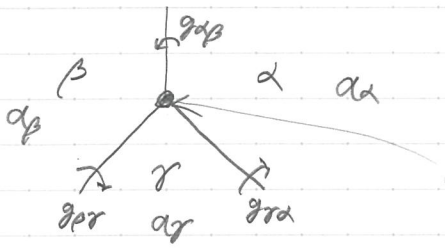
⇒ However, we can identify

Well-defined charge of $W(C)$ = #(Boxes) mod N

KOKUYO LOOSE-LEAF ノ-836BT 6 mm ruled×36 line

We want to construct a top. defect measuring #(Boxes) mod N

Data of gauge fields



codim-2 $= M_2$

$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$

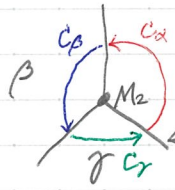
$U_k(M_2)$: On triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ including M_2,

$$\hat{g}_{\alpha\beta} \cdot \hat{g}_{\beta\gamma} \cdot \hat{g}_{\gamma\alpha} = e^{\frac{2\pi i}{N} k}$$

· We can check this op: $U_k(M_2)$ is topological

· $\langle U_k(M_2)\, W(C) \rangle = e^{\frac{2\pi i}{N} k \cdot \mathrm{Link}(M_2, C)} \langle W(C) \rangle$

($\mathrm{Link}(M_2, C)$: linking #)



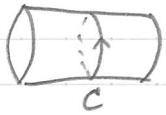
$C = C_\alpha + C_\beta + C_\gamma$

$g_{\gamma\alpha} \mapsto e^{\frac{2\pi i}{N}} g_{\gamma\alpha} = \hat{g}_{\gamma\alpha}$

$$= \mathrm{tr}\left\{ g_{\gamma\alpha} \left(P e^{i \int_{C_\gamma} a_\gamma}\right) g_{\beta\gamma} \left(P e^{i \int_{C_\beta} a_\beta}\right) g_{\alpha\beta} \left(P e^{i \int_{C_\alpha} a_\alpha}\right) \right\}$$

$$= e^{\frac{2\pi i}{N} k} W(C)$$

finite-T YM



Polyakov loop: $P(C) = \mathrm{tr}\left(\underbrace{P e^{i \oint_C a}}_{\Omega}\right)$

center sym. · $\mathbb{Z}_N$ sym : $\Omega \mapsto e^{\frac{2\pi i}{N}} \Omega$ 0-form sym in 3 dim

· $\mathbb{Z}_N$ 1-form sym is an extensions of this for generic Wilson lines on any 4 dim mfd

Connection formulas

$$a_\beta = g_{\beta\alpha}^{-1} a_\alpha g_{\beta\alpha} - i g_{\beta\alpha}^{-1} d g_{\beta\alpha}$$

$$g_{\alpha\beta} \longrightarrow C_{\alpha\beta} \cdot g_{\alpha\beta} = \widehat{g}_{\alpha\beta}$$

$C_{\alpha\beta}$: center element $= e^{\frac{2\pi i}{N} k_{\alpha\beta}}$

$\Rightarrow$ this replacement does not change connection formulas on all double overlaps.

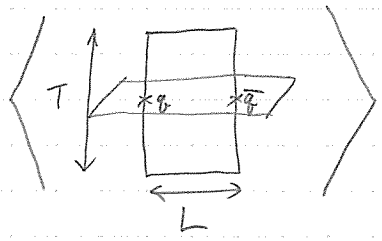
cf.) $\mathbb{Z}_N^{(1)}$ does not exist if field matter φ exists

$$\varphi_\beta = g_{\beta\alpha} \varphi_\alpha \longrightarrow e^{\frac{2\pi i}{N}} g_{\beta\alpha} \varphi_\alpha$$

$\rightarrow$ connection formula is affected by $U(1)_2$

Spontaneous Breaking

0-form sym. $\langle \varphi(x)^* \varphi(0) \rangle \xrightarrow[\text{vol}(X)\to\infty]{|x|\to\infty} \exists\, v^2 \neq 0$



$= \begin{cases} 0 & \text{unbroken} \\ e^{-\#(L+T)} & \text{(up-to perimeter law)} \\ & \text{spontaneously broken} \end{cases}$

$T, L \to \infty$
$\text{vol} \to \infty$

Low energy theorem for broken $\mathbb{Z}_N^{(1)} \to \mathbb{Z}_1^{(1)}$

$\Rightarrow$ Topological order

$$\#(\text{vac on } T^3 \times \mathbb{R}^{\text{time}}) = N^3$$
$$\#(\text{vac on } S^3 \times \mathbb{R}) = 1$$

$SU(N)$ YM $\to \mathbb{Z}_N^{(1)}$ — gauging this

3/5

YM @ $\theta = \pi$

CP — 't Hooft anom. ↓ anom. matching

: CP broken $\to$ 2 vacua

: $\mathbb{Z}_N^{(1)} \to \mathbb{Z}_{\frac{N}{2}}^{(1)}$ or smaller

$\rightsquigarrow \mathbb{Z}_2$ TQFT

9/29 Review of 1st day

Symmetry $\to$ Qualitative behavior of QFT

Ordinary sym $\to$ Transformation on local op. $\quad \langle \varphi(x) \rangle \begin{cases} \neq 0 & \text{SSB} \\ = 0 & \text{symmetric} \end{cases}$

Gauge theory — We don't have good "local" order parameters

Wilson - 't Hooft ; Instead, we can use "loop" op.

$\langle W(C) \rangle \sim \begin{cases} \text{Area} & : \text{Conf.} \\ \text{Perimeter} & : \text{Deconf.} \end{cases}$ ↺ phase separation

- We introduce a new notion of sym.
  $\mathbb{Z}_N$ 1-form sym.

$W(C) \mapsto e^{\frac{2\pi i}{N}} W(C)$ (for $SU(N)$ YM)

Higher form sym G

$U_g(M_{d-p-1})$: topological op. living on codim-$(p+1)$ mfd specified also by $g \in G$

Commutation relation $(G = \mathbb{Z}_N)$ $\quad (k \sim k+N)$

$U_k(S^{d-p-1}) W(C_p) = e^{\frac{2\pi i}{N} k \cdot q \cdot \text{Link}(C_p, S^{d-p-1})} \cdot W(C_p)$

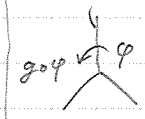
- $p=0 \longrightarrow$ Ordinary sym.
- $p \geq 1 \longrightarrow G$ = Abelian group $= \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots$

- for discrete sym., Gauge field = {Transition functions $g_{\alpha\beta}$}
  = Network of symmetry defect

If this gauge field is dynamical, (when G is 0-form sym.)

Pure G gauge theory = Dijkgraaf-Witten theory

$g_{\alpha\beta}$ = sym. defect



- Anomaly : d-dim QFT w/ sym G $\longrightarrow$ A: bg gauge field

$$Z[A+\delta_\Lambda A] = Z[A] \cdot e^{i\int \mathcal{A}[\Lambda, A]}$$

↑ If no local counter term does eliminate, we call it 't Hooft anom.

- Anomaly inflow

$\exists$ (d+1) dim topological theory $S_{d+1}[A]$,

$Z[A] \cdot e^{iS_{d+1}[A]}$ is gauge invariant

Plan

- Gauging $\mathbb{Z}_N^{(1)}$ (1-form sym.) of 4d SU(N) YM

  ↳ B: $\mathbb{Z}_N$ 2-form gauge field

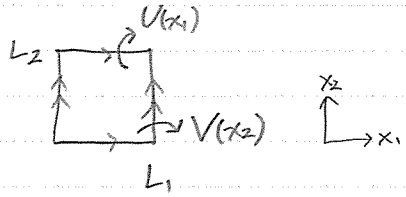
- Q_{top} = Integer + $\underbrace{\frac{N}{4\pi}\int B \wedge B}_{1/N \cdot \text{int}}$

- Applications to Dashen's CP-broken phase at $\theta = \pi$

't Hooft twisted BC and $\mathbb{Z}_N^{(1)}$-background

$M_4 = T^4$: 4-torus

φ: a matter field in a rep. ρ of $SU(N)$ gauge group



connection formulas for φ : $\begin{cases} \varphi(x_1+L_1, x_2) = \rho(V(x_2))\,\varphi(x_1,x_2) \\ \varphi(x_1, x_2+L_2) = \rho(U(x_1))\,\varphi(x_1,x_2) \end{cases}$

Cocycle condition (= consistency condition)

$$\begin{aligned} \varphi(L_1,L_2) &= \rho(V(L_2))\,\varphi(0,L_2) \\ \| \quad\quad &= \rho(V(L_2))\,\rho(U(0))\,\varphi(0,0) \end{aligned}$$

$$\begin{aligned} \varphi(L_1,L_2) &= \rho(U(L_1))\,\varphi(L_1,0) \\ &= \rho(U(L_1))\,\rho(V(0))\,\varphi(0,0) \end{aligned}$$

$$\Rightarrow \rho(V(L_2)V(0)) = \rho(V(L_1)V(0))$$

$\rightsquigarrow$ If φ is a fund. matter, $\quad V(L_2)U(0) = U(L_1)V(0)$

If φ is an adj. matter, $\quad V(L_2)V(0) = U(L_1)V(0)\, e^{\frac{2\pi i}{N} n_{12}}$

This is nothing but the insertion of sym. generators of $\mathbb{Z}_N^{(1)}$

※ If ρ = fund., $\quad \varphi(x_1+L_1, x_2) = V(x_2)\,\varphi(x_1,x_2)$
↑ N-dim vector

ρ = adj., $\quad \varphi(x_1+L_1, x_2) = V(x_2)\,\varphi(x_1,x_2)\,V(x_2)^{-1}$
↑ N×N matrix

◦ $SU(N)$ gauge theory VS. $\frac{SU(N)}{\mathbb{Z}_N}$ gauge theory

· For $SU(N)$ gauge theory,
You should take $n_{ij} = 0 \mod N$ in path int.

· For $\frac{SU(N)}{\mathbb{Z}_N}$ gauge theory, you have to sum up $\forall n_{ij} \overset{=0\sim N-1}{}$ in path int.
$\uparrow \sum_{\{n_{ij}\}} \int Da$

· As $\{n_{ij}\}$ only takes discrete values, they don't affect local propagating modes. (no kinetic term)

· What is different in the set of contents of genuine line operators

For $SU(N)$ theory

$W(C)$ ← genuine-line op.
$H(C,S)$ ← boundary of topological surface op.
↑
't Hooft loop ($\partial S = C$)

For $\left(\frac{SU(N)}{\mathbb{Z}_N}\right)_0$ theory,

$W(C,S)$ ← no longer a genuine line, but a bdry of top. surface.
$H(C)$ ← genuine line op.

(cf. Aharony, Seiberg, Tachikawa)

Kapstin, Seiberg '14

Wilson - 't Hooft criterion

Conf. { W ~ Area, H ~ Perimeter

Deconf. { W ~ Perimeter, H ~ Area

Gapped phase of SU(N) theory

{ Conf. Topological trivial — #(vac) = 1 on any spatial mfd
{ Deconf. $\mathbb{Z}_N$ TQFT — { #(vac on $S^3 \times \mathbb{R}$) = 1, #(vac on $T^3 \times \mathbb{R}$) = N^3

Gapped phase of $\frac{SU(N)}{\mathbb{Z}_N}$ theory

{ Conf. $\mathbb{Z}_N$ topological — { #(vac S^3) = 1, #(vac T^3) = N^3
{ Deconf. Trivial — #(vac) = 1 on any spatial mfd

$$\Rightarrow Q_{top} = (\text{integer}) + \frac{1}{N}\varepsilon_{ijkl}\, n_{ij} n_{kl}$$

fraction shift of top. charges

(van Baal '80s)

We introduce a convenient continuum description for $\mathbb{Z}_N$ 2-form gauge field (Kapustin Seiberg '14)

e.g.) SQM w/ N wells

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - g\cos(N\varphi) + i\frac{\theta}{2\pi}\dot{\varphi} \qquad \varphi \sim \varphi + 2\pi$$

sym. $\mathbb{Z}_N : \varphi \to \varphi + \frac{2\pi}{N}$ $\qquad (\theta = 2\pi \cdot S)$

$C : \varphi \to -\varphi$ $\quad (\theta = 0 \text{ or } \pi)$

We try to gauge $\mathbb{Z}_N$

$\begin{cases} A : U(1) \text{ gauge field} \\ \Phi : 2\pi\text{-periodic scalar} \end{cases}$ ⇆ bg field

We put a constraint $NA = d\Phi$

Motivation $e^{i\int_0^p A} = e^{i\frac{1}{N}\int_0^p d\Phi} = e^{i\frac{1}{N}(\overbrace{\Phi(p)-\Phi(0)}^{2\pi n}\ (n\in\mathbb{Z}))}$

$= e^{i\frac{2\pi}{N}n}$

($\mathbb{Z}_N$ transition function)

$(\mathrm{tr}(\hat{g}\, e^{-\beta H}))$

We couple (A, Φ) to $\mathcal{L}$ to gauge its $\mathbb{Z}_N$

We postulate the gauge inv.

$\begin{cases} A \mapsto A + d\lambda \\ \Phi \mapsto \Phi + N\lambda \end{cases}$ $\qquad \varphi \mapsto \varphi + \lambda$

Gauged $\mathcal{L}$: $\mathcal{L} = \frac{1}{2}(\dot{\varphi} - A_0)^2 - g\cos(N\varphi - \Phi) + i\frac{\theta}{2\pi}(\dot{\varphi} - A_0)$

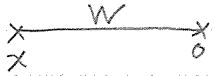
This is the theory w/ $\mathbb{Z}_N$ gauge redundancy

Sanity check

Original local op. $\dot{\varphi}$, $e^{i\varphi}$

Local op. in gauged theory $\left((e^{i\varphi})^N \sim\right) e^{iN\varphi}\, e^{-i\Phi}$

$e^{i\varphi(x)}\, e^{i\int_0^x A}\, e^{-i\varphi(0)}$



It looks like the consistency with $\mathbb{Z}_N$ gauge inv.

Now, we go back to 4d SU(N) YM

We try to gauge its $\mathbb{Z}_N^{(1)}$ sym.

$$\begin{cases} B : U(1)\ 2\text{-form gauge field} & (= \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu) \\ C : U(1)\ 1\text{-form gauge field} & (= C_\mu dx^\mu) \end{cases}$$

We put the constraint $NB = dC$

This constraint respects the 1-form gauge inv.

$$\begin{cases} B \mapsto B + d\Lambda^{(1)} \\ C \mapsto C + N\Lambda^{(1)} \end{cases} \leftarrow U(1) \text{ gauge field}$$

We couple (B, C) to SU(N) YM

We embed the SU(N) gauge field a into U(N) gauge field $\hat{a}$

$$\hat{a} = \text{“}\, a + \frac{1}{N} C \mathbb{1}_{N\times N} \text{”} \quad \leftarrow \text{just a motivation}$$

More regorously, we perform $\int \mathcal{D}\hat{a}\big|_{\mathrm{tr}(\hat{a}) = C}$

$\hat{a}$ is affected by 1-form gauge trans. $\quad \hat{a} \mapsto \hat{a} + \Lambda^{(1)} \mathbb{1}_{N\times N}$

SU(N) field strength : $f = da + ia^2$

$\downarrow$

U(1) : $\quad \hat{f} = d\hat{a} + i\hat{a}^2$

However, $\hat{f}$ is not gauge cov. under 1-form trans.

$$\hat{f} \mapsto \hat{f} + d\Lambda^{(1)} \mathbb{1}_{N\times N}$$

Gauge-cov combination : $f - B\mathbb{1}_{N\times N}$

$\Downarrow$

Gauged $\mathcal{L}$: $\mathcal{L} = \frac{1}{g^2} \mathrm{tr}\left((\hat{f}-B) \wedge *(\hat{f}-B)\right) + \frac{i\theta}{8\pi^2} \mathrm{tr}\left((\hat{f}-B)^2\right)$

$$Z[B] = \int \mathcal{D}\hat{a}\big|_{\mathrm{tr}(\hat{a}) = C} e^{-S_{\text{gauged}}[\hat{a}, B]}$$

・$PSU(N)_k$ theory $\left(PSU(N) = \frac{SU(N)}{\mathbb{Z}_N}\right.$, Projection $\left.SU(N)\right)$

$$Z_{PSU(N)_k} = \int \mathcal{D}B\, Z[B] \cdot e^{i\frac{kN}{4\pi}\int B\wedge B}$$

※ $NB = dC$ より, $dB = 0$
⇒ B は propagate しない

・Wilson loop is no longer gauge inv.

$$W(C) = \mathrm{tr}\left(P e^{i\oint a}\right) \longrightarrow "W(C) = \mathrm{tr}\left(P e^{i\oint \tilde{a}}\right)"$$

$\tilde{a} \mapsto \tilde{a} + \Lambda^{(1)}$

$$\Rightarrow W(C) \mapsto e^{i\oint \Lambda^{(1)}} W(C)$$

"correct" def. of $W(C)$ in $PSU(N)$ theory

$$W(C, S) = \mathrm{tr}\left(P e^{i\oint_C \tilde{a}}\right) \cdot \underbrace{e^{-i\int_S B}}_{\text{topological surface}}$$

@T4, "$e^{\frac{2\pi}{N} i n_{ij}} = e^{i\int_{\Sigma_2} B}$"

Fractional top. charge

$$Q_{top}[B] = \frac{1}{8\pi^2}\int \mathrm{tr}(\tilde{f} - B)^2 \qquad \mathrm{tr}(\tilde{f}) = \mathrm{tr}(d\tilde{a} + i\tilde{a}^2) = d(\underbrace{\mathrm{tr}\,\tilde{a}}_{=C})$$

$$= \frac{1}{8\pi^2}\int \left\{\mathrm{tr}(\tilde{f}^2) - 2\underbrace{\mathrm{tr}(\tilde{f})}_{NB}\wedge B + NB^2\right\}$$

$$= \frac{1}{8\pi^2}\int \mathrm{tr}(\tilde{f}^2) - \frac{N}{8\pi^2}\int B\wedge B$$

$\frac{1}{8\pi^2}\int \mathrm{tr}(\tilde{f}^2)$ = instanton # of $U(N)$ bundle $\in \mathbb{Z}$

$\frac{N}{8\pi^2}\int B\wedge B = \frac{1}{N}\cdot \underbrace{\frac{1}{2}\int \frac{\widetilde{NB}}{2\pi}\wedge\frac{NB}{2\pi}}_{\text{instanton \# of } U(1) \text{ bundle} \in \mathbb{Z}} \in \frac{1}{N}\mathbb{Z}$ $\quad (NB = dC)$

$\widetilde{NB} = dC$

(continuum derivation of van Baal)

Mixed anom. bet $\mathbb{Z}_N^{(1)}$ and CP @ $\theta=\pi$

YM @ $\theta=\pi$ $\Rightarrow$ CP sym.

$$i\frac{\theta}{8\pi^2}\int \mathrm{tr}(f^2) \xrightarrow{CP} -i\frac{\theta}{8\pi^2}\int \mathrm{tr}(f^2)$$

$$\downarrow @\theta=\pi$$

$$= i\frac{\pi}{8\pi^2}\int \mathrm{tr}(f^2) - i2\pi\cdot \underbrace{\frac{1}{8\pi^2}\int \mathrm{tr}(f^2)}_{\in\mathbb{Z}}$$

Let us check if CP still exists under B

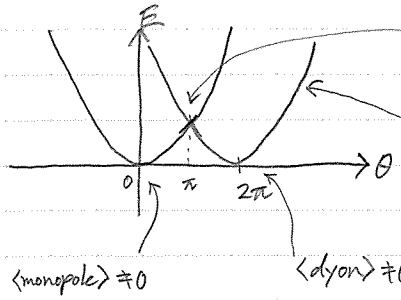
$$Z[B] = \int \mathcal{D}\tilde{a}\, e^{-\frac{1}{g^2}|\hat{f}-\hat{B}|^2 + i\frac{\pi}{8\pi^2}\mathrm{tr}(\hat{f}-B)^2}$$

$$\xrightarrow{CP} \int \mathcal{D}\tilde{a}\, e^{-\frac{1}{g^2}|\hat{f}-\hat{B}|^2} e^{i\frac{\pi}{8\pi^2}\mathrm{tr}(\hat{f}-B)^2} \underbrace{e^{-i\frac{2\pi}{8\pi^2}\mathrm{tr}(\hat{f}-B)^2}}_{=}$$

$$e^{-2\pi i\cdot\left\{\underbrace{\frac{1}{8\pi^2}\int \mathrm{tr}(f)^2}_{\in\mathbb{Z}} - i\frac{N}{8\pi^2}\int B^2\right\}}$$

$$= e^{\frac{iN}{4\pi}\int B^2}$$

$$\therefore Z[B] \xrightarrow{CP} \underbrace{e^{i\frac{N}{4\pi}\int B^2}}_{\text{anomalous phase}} \cdot Z[B]$$



CP is spontaneously broken @ $\theta=\pi$

dyon : $(e,m)=(-1,1)$ @ $\theta=0$

electric charge

$e = \frac{\theta}{2\pi}$ due to (Witten-effect)

monopole Coulomb energy $\sim e^2+g^2$

$\sim \left(\frac{\theta}{2\pi}\right)^2+g^2$

dyon charge @θ $\to \left(-1+\frac{\theta}{2\pi}, g\right)$

Coulomb energy $\sim \left(-1+\frac{\theta}{2\pi}\right)^2+g^2$

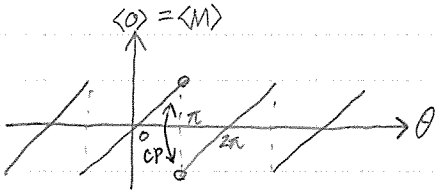
$\Rightarrow$ Anomaly at least explains that YM @$\theta=\pi$ must be

- ~~CP~~
- $\mathbb{Z}_n$ TQFT
- gappless (Coulomb phase)

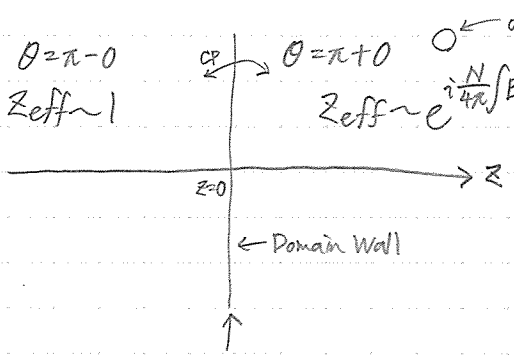
(このどれかは満たす筈)

Order parameter

$\langle O\rangle = \langle \varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}\rangle$



CP : $\langle O\rangle \mapsto -\langle O\rangle$



← area law : Bulk conf.

This is a non-trivial SPT phase

$SU(N)_1$ C.S theory — Deconf. at bdry

9/30 2 dim nonlinear σ model : Toy model of YM

· Asymptotic freedom → Dynamical mass gap generation

· (for vector-like σ model,) We can solve this model analytic
↑ # of DOF ~ N (mean-field become exact) in large N limit.

For cond-mat people, this is a good theory for anti-ferro spin system.

◦ 2 dim $\mathbb{CP}^{N-1}$ σ-model

EOM of a_μ : $a_\mu = -i\bar{z}\partial_\mu z$

$$S = \frac{1}{g}\int |(\partial_\mu + i a_\mu) z^i|^2 + \frac{i\theta}{2\pi}\underbrace{\int da}_{\text{instanton \# in 2 dim}}$$

$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$: N-component cpx scalar

constraint : $\vec{z}^{\dagger}\cdot\vec{z} = 1$

a : U(1) gauge field (coupled to U(1) sym. $z_i \mapsto e^{id} z_i$)

Symmetry · spin-rotation sym.

$\frac{SU(N)}{\mathbb{Z}_N}$: $\vec{z} \mapsto \overset{SU(N)}{U}\cdot\vec{z}$

$\mathbb{Z}_N = \{ \mathbb{1}_N, e^{\frac{2\pi i}{N}}\mathbb{1}_N, \cdots, e^{\frac{2\pi i}{N}(N-1)}\mathbb{1}_N \} \subset SU(N)$

$\vec{z} \mapsto \underbrace{e^{\frac{2\pi i}{N}}}\vec{z}$
overall U(1) phase rot. of $\vec{z}$

However, we have gauge sym. $\vec{z} \mapsto e^{i\alpha}\vec{z}$

and $\mathbb{Z}_N$ is just a special $(\alpha = \frac{2\pi}{N})$

↑ This is included inside the gauge redundancy

(any gauge-inv op, $(z^*)^n \times z^n \rightsquigarrow$ neutral in $\mathbb{Z}_N \subset SU(N)$
⇒ faithful rep. is realized by $\frac{SU(N)}{\mathbb{Z}_N}$ (not by $SU(N)$)

C : charge conj. $(\theta = 0 \text{ or } \pi)$

$C : \begin{cases} z \mapsto z^* \\ a \mapsto -a \end{cases}$ $\quad |(\partial_\mu + ia_\mu)z|^2 \mapsto |(\partial_\mu - ia_\mu)z^*|^2 = |(\partial_\mu + ia_\mu)z|^2$

↑ kinetic term is obviously inv.

θ-term flips its spin : $\frac{i\theta}{2\pi}\int da \xrightarrow{C} -\frac{i\theta}{2\pi}\int da$

$\theta \sim \theta + 2\pi \Rightarrow \theta = 0$ and $\theta = \pi$ are C-sym. points

• 2d $\mathbb{CP}^1$ σ-model is a low energy description of 1d quantum anti-ferro. Heisenberg model $(|S| >> 1)$

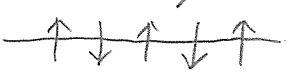
↑ size of spin

(See Frankin's textbook) $\rightsquigarrow$ Haldane conj.

$\hat{H} = J\sum_{i=1}^{L} \hat{S}_i \cdot \hat{S}_{i+1}$ $\quad (J > 0)$: XXX model

↑ site

Classical ground state — anti-ferro.

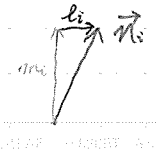


↳ path-int rep. for each $(\vec{S}_i = S\vec{n}_i)$

↑ unit vector

$\vec{n}_i = (-1)^i \cdot \underline{\vec{m}_i} + \underline{\vec{l}_i}$

Large wavelength ← low wavelength, small flac.



$$S_{eff}[\vec{m}] = \frac{1}{g}\int |d\vec{m}|^2 + \frac{\theta}{2\pi}\frac{1}{4}\int \vec{m}\cdot(d\vec{m}\times d\vec{m})$$

O(3) nonlinear σ-model

↑ this term come out of the WZ term ($\int d\varphi\,(1-\cos\theta)$)

$\theta = 2\pi S$ — $\begin{cases} S: \text{int} & \Rightarrow \theta = 0 \\ S: \text{half-int} & \Rightarrow \theta = \pi \end{cases}$

$g \sim \frac{1}{S}$ ← If $|S| \gg 1$, we have a reliable weak-coupling @ high energy

Let us introduce a spinon field : $\vec{m} = z^* \vec{\sigma} z$

↑ Pauli-matrix $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ ↑ cpx scalar

$\vec{m}^2 = 1 \Leftrightarrow z^*\cdot z = 1$

$a_\mu = -i z^* \partial_\mu z$

Decomp. of $\vec{m}$ into $\vec{z}$ has a U(1) ambiguity

U(1)-gauge redundancy

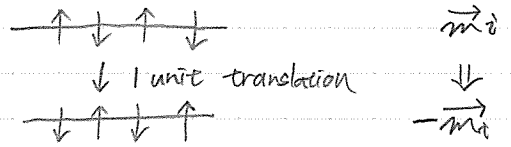
$$S_{eff}[\vec{m}] \Rightarrow S_{CP^1} = \frac{1}{g}\int |(\partial_\mu + i a_\mu) z|^2 + i\frac{\theta}{2\pi}\int da$$

Origin of sym.

· Spin rot SO(3) : $\vec{S}_i \to O\cdot\vec{S}_i$ (O ∈ SO(3))

$= SU(2)/Z_2$: $\vec{z} \to U\vec{z}$ (U ∈ SU(2)) $U \sim -U$

· Charge conj. = lattice translation by 1 unit



$\vec{m} = z^*\vec{\sigma}z$

↓

$-\vec{m}$, $z \mapsto (i\sigma_2)z^*$

int. S half int. S

$\theta = 0$ or π is protected

in derivation of low energy QFT

(Haldane '83)

Haldane conj. · For int S, Heisenberg spin is gapped ← AKLT ↓ solvable
↙ $S=\frac{1}{2}$: Bethe ansatz

· For half int S, Heisenberg spin chain is gapless ↑ '88 Affleck, Haldane

(For SU(N) spin, 2 dim flag mfd σ model)

$SU(2)_1$ WZW CFT

Anomaly of 2d $\mathbb{CP}^{N-1}$ σ model @ $\theta=\pi$

(今日は SU(N) はやらない、$\mathbb{CP}^{N-1}$ のみ ※large N は取らない)

⇝ Mixed anom. between $\frac{SU(N)}{\mathbb{Z}_N}$ and C

← We introduce a bg gauge field A

$$Z[A] \xmapsto{C} e^{i\pi} Z[A]$$

↑ anomaly

bg $\frac{SU(N)}{\mathbb{Z}_N}$ gauge field : → $\begin{cases} \hat{A} : U(N) \text{ 1-form gauge field} \\ B : U(1) \text{ 2-form gauge field} \end{cases}$

$NB = d(\mathrm{tr}\hat{A})$

(※昨日 (4 dim) A : SU(N) gauge field ⇝ $\mathbb{Z}_N$ 2-form gauge field (B : U(1) 2-form, C : U(1) 1-form, w/ $NB = dC$)

"local, $\hat{A} = A + \frac{1}{N} C \mathbb{1}_N$" ⇝ $\mathrm{tr}\hat{A} = C$

↑ これと同じ)

1-form gauge transformation $\begin{cases} B \mapsto B + d\lambda \\ \hat{A} \mapsto \hat{A} + \lambda \mathbb{1}_N \end{cases}$ this is consistent w/ the constraint

$(\partial_\mu + i a_\mu) z \Rightarrow (\partial_\mu + i a_\mu + i\hat{A}_\mu) z$

↑ cov. derivative of z w/ bg

for 1-form gauge inv. of $D_\mu z$, $a \mapsto a - \lambda$

What about θ-term? $\frac{1}{2\pi} da \xrightarrow{\text{1-form trans.}} \frac{1}{2\pi} da - \frac{1}{2\pi} d\lambda$

→ $\frac{1}{2\pi}(da + B)$ is the correct combination for gauge inv.

$$S_{\text{gauged}} = \frac{1}{g}\int |(\partial_\mu + i a_\mu + i\hat{A}_\mu) z|^2 + \frac{i\theta}{2\pi}\int (da + B)$$

$$Z[\hat{A}, B] = \int \mathcal{D}a\,\mathcal{D}z^*\mathcal{D}z\; e^{-S_{\text{gauged}}}$$

· w/ $(\hat{A}, B)$, θ is no longer 2π-periodic

$$S_{\text{gauged}} \xrightarrow{\theta\mapsto\theta+2\pi} S_{\text{gauged}} + i\frac{2\pi}{2\pi}\int (da + B)$$

(da: 2π quantized)

$$Z_{\theta+2\pi}[\hat{A}, B] = e^{-i\int B}\, Z_\theta[\hat{A}, B]$$

$$NB = d(\mathrm{tr}\hat{A}) \;\to\; \int B = \frac{1}{N}\int \underbrace{d(\mathrm{tr}\hat{A})}_{2\pi\text{ quantized}} \qquad \in \frac{2\pi}{N}\mathbb{Z}$$

$$C:\; Z_{\theta=\pi}[\hat{A}, B] \xrightarrow{C} Z_{\theta=-\pi}[\hat{A}, B] = e^{i\int B}\, Z_{\theta=\pi}[\hat{A}, B]$$

($\theta = \pi - 2\pi$; $e^{i\int B}$: anom. phase)

· for even N, we don't have a local counterterm to eliminate this phase.

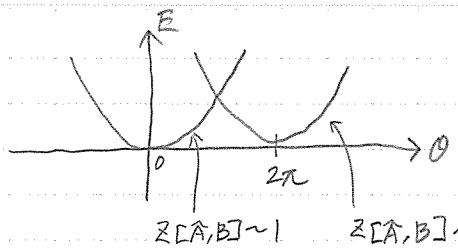
(N ↔ $\mathbb{CP}^{N-1}$ の N)

(for odd N, the counterterm is exist.)

We have found a mixed anom. between $\frac{SU(N)}{\mathbb{Z}_N}$ and C @ $\theta=\pi$

Anom. matching requires

- Spontaneous C-breaking @ $\theta=\pi$ (this is the case for $N\geq 3$) — gapped. (in odd N, consider 1-form sym.)
- Gapped excitations w/o sym. breaking (this is the case for $N=2$) ← $SU(2)_1$ WZW
- ~~SSB of PSU(N)~~ ← prohibited by Coleman-Mermin-Wagner (1+1 dim field theory で continuum sym. は SSB しない)



↑ nontrivial on PSU(N) protected SPT phase

2d SU(N) WZW model

(N: 商の CP^{N-1} のNとは別のパラメータ)

V : SU(N) valued field

$\frac{SU(N)\times SU(N)}{Z_N}$ chiral symmetric theory

⇝ principal chiral model $\quad S = \frac{1}{g}\int \mathrm{tr}(\partial_\mu V^\dagger \cdot \partial_\mu V)$

WZW model = principal chiral + 2d WZW term

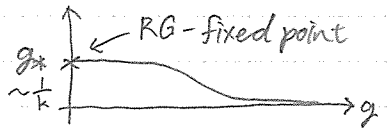
$$S = \frac{1}{g}\int_X \mathrm{tr}(\partial_\mu V^* \cdot \partial_\mu V) + i\frac{k}{12\pi}\int_{Y_3} \mathrm{tr}((V^\dagger dV)^3)$$

(k: WZW level, $\partial Y_3 = X$)

$SU(N)\times SU(N)$ sym. : $V \mapsto L^\dagger V R$

$\Downarrow$

(L, R)

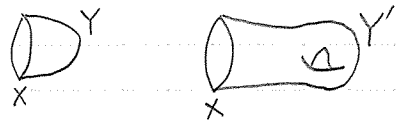


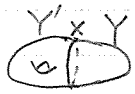
⇒ scale inv. theory

- $k \in \mathbb{Z}$ WZW term could be ambiguous

WZ term is topological : $d\left(\frac{1}{12\pi}\mathrm{tr}(V^\dagger dV)^3\right) \sim \mathrm{tr}((V^\dagger dV)\wedge(V^\dagger dV)^3)$

$= -\underbrace{\mathrm{tr}((V^\dagger dV)^3 \wedge (V^\dagger dV))}_{=0}$



$WZ_Y \neq WZ_{Y'}$  ← closed 3dim mfd

$$WZ_Y - WZ_{Y'} = \frac{1}{12\pi}\int_{Y-Y'} \mathrm{tr}\left(V^\dagger dV\right)^3 \qquad \in 2\pi\mathbb{Z}$$

$$= 24\pi^2 \times \text{winding \# of } SU(N), \qquad \pi_3(SU(N)) = \mathbb{Z}$$

$N=2$ $\quad \dfrac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2} \supset \dfrac{SU(2)_{L+R}}{\mathbb{Z}_2} \times (\mathbb{Z}_2)_A$

$\dfrac{SU(2)_{L+R}}{\mathbb{Z}_2} = SO(3)$ spin $\qquad (\mathbb{Z}_2)_A = (\mathbb{Z}_2)_C$

We consider an explicit breaking term, $\dfrac{SU(2)\times SU(2)}{\mathbb{Z}_2} \to \dfrac{SU(2)}{\mathbb{Z}_2} \times (\mathbb{Z}_2)_A$

$V_{\text{perturbation}} = g'|\mathrm{tr}\,V|^2$

(Under $SU(2)\times SU(2) \ni (L,R)$, $\mathrm{tr}\,V \mapsto \mathrm{tr}(L^\dagger V R) \neq \mathrm{tr}\,V$ for generic)

$(L,L) \in SU(2)_{\text{diag}}$: $\mathrm{tr}\,V \mapsto \mathrm{tr}(L^\dagger V L) = \mathrm{tr}\,V$

$(-\mathbb{1}, \mathbb{1}) \in (\mathbb{Z}_2)_A$ $\quad \mathrm{tr}\,V \mapsto -\mathrm{tr}\,V$, $(\mathrm{tr}\,V)^2 \mapsto (\mathrm{tr}\,V)^2$ $\quad \therefore$ O.K.

$g' \to +\infty$

$SU(2)_k$ WZW $\to$ 2d CP^1 σ-model @ $\theta = \pi k$

$\Rightarrow \mathrm{tr}\,V = 0$

Such a $V \in SU(2)$ can be parametrized as

$$V = W^\dagger \left\{ i \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\} W$$

(W ← $SU(2)$ matrix)

$$W = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1 \end{pmatrix} = \vec{z}\,(i\sigma_2)\,\vec{z}^*$$

$(\mathbb{Z}_2)_A = (\mathbb{Z}_2)$ lattice translation

$$S_{WZW,k}[V = \{W^{-1}i(\partial - \bar{\partial})W\}]$$

$$= \cdots$$

$$= \frac{\#}{g}\int |(\partial_\mu + i a_\mu) z|^2 + i\frac{\pi k}{2\pi}\int da$$

($a_\mu = -i\bar{z}^*\partial_\mu z$)

$SU(2)_k$ WZW

For odd k $\Rightarrow$ mixed anom. for $\frac{SU(2)}{\mathbb{Z}_2} \times \mathbb{Z}_2$ $\left(C\ \frac{SU(2)\times SU(2)}{\mathbb{Z}_2}\right)$

$\rightsquigarrow$ nontrivial low energy behavior

For even k $\Rightarrow$ No anom.

$\Rightarrow$ $|\mathrm{tr}V|^2$ deformation gets a mass gap

Comment.

- $SU(2)_1$ WZW CFT does not have a relevant deformation keeping $\frac{SU(2)}{\mathbb{Z}_2} \times \mathbb{Z}_2$

(g' ; irrelevant part $\Rightarrow g'|\mathrm{tr}V|^2 \xrightarrow{IR} 0$ going back $SU(2)_1$ WZW CFT)

- For $k \geq 2$, $SU(2)_1$ WZW, $|\mathrm{tr}V|^2$ is relevant perturbation

$$\Rightarrow \begin{cases} \text{even } k : SU(2)_1 + |\mathrm{tr}V|^2 \rightarrow \text{mass gap} \\ \text{odd } k : SU(2)_1 + |\mathrm{tr}V|^2 \rightarrow \begin{cases} \varnothing \\ SU(2)_1 \text{ WZW} \end{cases} \end{cases}$$

Extension to SU(N) WZW

$$\frac{1}{g}\mathrm{tr}(\partial_\mu V^\dagger \partial_\mu V) + i\frac{k}{12\pi}\mathrm{tr}\left((V^\dagger dV)^3\right)$$

$$\frac{SU(N)_L \times SU(N)_R}{\mathbb{Z}_N} \supset \frac{SU(N)_V}{\mathbb{Z}_N} \times (\mathbb{Z}_N)_A$$

motivation : SU(N) ver. anti-ferro
1 unit lattice trans. $\longrightarrow$ $(\mathbb{Z}_N)_{\text{lattice translation}}$

WZW + double trace deformation :

$$g_1|\mathrm{tr}V|^2 + g_2|\mathrm{tr}V^2|^2 + \cdots + g_{\frac{N}{2}}|\mathrm{tr}V^{\frac{N}{2}}|^2$$

$g_{i=1,2,\cdots,\frac{N}{2}} \to \infty$ $\quad (V = W(1,-1)W^\dagger$ for $SU(2))$

$$V = W\begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} W^\dagger$$

$S_{WZW}\left[V = W(1,\omega,\cdots,\omega^{N-1})W^\dagger\right]$

$\Rightarrow$ flag-mfd σ model w/ target space $\frac{SU(N)}{U(1)^{N-1}}$

Fact $\Rightarrow$ Mixed anom. bet. $\left(\frac{SU(N)}{\mathbb{Z}_N}\right)_V \times (\mathbb{Z}_N)_A$

· Application of anom.

SU(N) generalization of Haldane conj.

· SU(N) anti-ferro w/

If $\gcd(N,P) = 1$ $\Rightarrow$ SU(N)$_1$ WZW
If $\gcd(N,P) \neq 0,1$ $\rightsquigarrow$ mass gap w/ degenerate vacua
If $\gcd(N,P) = 0$ $\Rightarrow$ Gapped trivial phase
$\uparrow P \bmod N$

Comment

· (2+1) dim anti-ferro
↖ quantum critical point (Deconfined criticality)

For square lattice
$\rightsquigarrow \left(\frac{SU(2)}{Z_2}\right)_{spin} \times \underbrace{U(1)_{top.}}_{\text{↖ magnetic monopole}}$ has an mixed anom.